Mixed mode (I + II) SIF calculation using surface displacements

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Chapter 1

Theory
1.1 Fundamental equations

Let’s assume a body lying in complex plane $Z$. The coordinate of any point in this plane can be expressed through a complex variable $z = x_1 + ix_2$. Displacements and stresses can be expressed through two analytical functions

$$
\phi = \phi(z); \quad \psi = \psi(z) \quad (1.1)
$$

as follows (Muskhelishvili, 1977):

$$
2\mu(u_1 + iu_2) = \kappa\phi(z) - z\phi'(z) - \psi(z) \quad (1.2)
$$

$$
\sigma_{11} + i\sigma_{12} = \phi'(z) + \phi'(z) - z\phi''(z) - \psi'(z) \quad (1.3)
$$

$$
\sigma_{22} - i\sigma_{12} = \phi'(z) + \phi'(z) + z\phi''(z) + \psi'(z) \quad (1.4)
$$

where

$$
\mu = \frac{E}{2(1 + \nu)} \quad (1.5)
$$

$$
\kappa = 3 - 4\nu \quad \text{(plane strain);} \quad \kappa = \frac{3 - \nu}{1 + \nu} \quad \text{(plane stress)} \quad (1.6)
$$

$E$ - the Young’s modulus, $\nu$ - the Poisson’s ratio, $u_i$ - the displacements and $\sigma_{ij}$ - the stresses, $i, j = 1, 2$. The overbar denotes complex conjugate.
1.2 Conformal mapping

The boundary conditions on a centre-crack in an infinite plate, that the normal and the shear stresses are zero, can be expressed more easily in another plane \( S \) which has a complex variable \( \zeta \) related to \( z \) through a complex valued function \( \omega \):

\[
z = \omega(\zeta).
\]  

(1.7)

The function

\[
z = \omega(\zeta) = R \left( \zeta + \frac{m}{\zeta} \right)
\]

(1.8)

where \( m \) is a shape parameter, \( 0 \leq m \leq 1 \), and \( R \) is a scale parameter maps a unit circle, \( \zeta = e^{i\theta} \), onto an elliptical contour, \( L \), in plane \( Z \). If \( m = 0 \) then \( L \) is a circle of radius \( R \). If \( m = 1 \) then \( L \) is a mathematical crack of zero thickness and length \( 4R \).
1.3 Fourier series representation

The analytical functions:

\[ \phi(\zeta) = \sum_{-\infty}^{+\infty} a_k \zeta^k; \quad \text{and} \quad \psi(\zeta) = \sum_{-\infty}^{+\infty} b_k \zeta^k \]  

(1.9)

The conformal mapping function:

\[ \Omega(\zeta) = \frac{\omega(\zeta)}{\omega'(\eta)} = \sum_{-\infty}^{+\infty} c_k \eta^k \]  

(1.10)

where \( a_k, b_k \) and \( c_k \) are complex coefficients.

For the centre crack, using (1.8):

\[ c_k = \begin{cases} 
  m & k = -1 \\
  (m^2 + 1)m^{k-1} & k = 2n + 1; \quad n = 0, 1, 2 \ldots \\
  0 & \text{all other } k 
\end{cases} \]  

(1.11)

for \( m < 1 \) (Sokolnikoff and Sokolnikoff, 1941).
1.4 Boundary conditions

The boundary conditions, zero normal and shear stresses on the crack contour, lead to:

\[ b_k = -\bar{a}_{-k} - \sum_l l a_l \bar{c}_{l-k-1} \]  \hspace{1cm} (1.12)

Substituting (1.9)–(1.12) into (1.2) one obtains:

\[ \kappa \sum_k a_k \zeta^k - \Omega \sum_k k \bar{a}_k \bar{\zeta}^{k-1} + \]
\[ + \sum_k \left( a_k + \sum_l l \bar{a}_l c_{l+k-1} \right) \bar{\zeta}^{-k} = 2\mu(u_1 + iu_2) \]  \hspace{1cm} (1.13)

If one introduce

\[ a_k = \alpha_k + i\beta_k \]  \hspace{1cm} (1.14)

then (1.13) will split into two real linear equations with respect to infinite number of real unknowns, \( \alpha_k = \text{Re}a_k \) and \( \beta_k = \text{Im}a_k \).
1.5 System of linear equations

Complex eqn (1.13) applied to $p$ points $\zeta$ leads to the system of $2p$ real equations:

$$
\sum_{k=-N}^{N} A_{jk}^i \alpha_k + \sum_{k=-N}^{N} B_{jk}^i \beta_k = 2\mu u_{1j}^i 
$$  \hspace{1cm} (1.15)

$$
\sum_{k=-N}^{N} C_{jk}^i \alpha_k + \sum_{k=-N}^{N} D_{jk}^i \beta_k = 2\mu u_{2j}^i 
$$  \hspace{1cm} (1.16)

where $j = 1, 2, \ldots p$, $u_{1j}^i$ and $u_{2j}^i$ are displacements at point $j$, and $A_{jk}^i$, $B_{jk}^i$, $C_{jk}^i$ and $D_{jk}^i$ are coefficients.

If $p > 2N + 1$ then the number of unknowns is smaller than the number of equations and the system is overdetermined. The solution can be found in the linear least squares sense, e.g. by QR decomposition (Golub and Van Loan, 1996).
1.6 Stress intensity factors

It was shown by Sih et al. (1962) that the complex stress intensity factor

\[ K = K_I - iK_{II} \]  \hspace{1cm} (1.17)

can be found as

\[ K = 2\sqrt{2}\pi \lim_{\zeta \to 1} \phi' (\zeta) \frac{\sqrt{\omega (\zeta) - \omega (1)}}{\omega' (\zeta)}. \]  \hspace{1cm} (1.18)

Note \( \sqrt{\pi} \) factor which does not appear in Sih et al. (1962).

Using (1.8) with \( m = 1 \) and (1.9) one obtains:

\[ K = 2\sqrt{\pi} \frac{\pi}{a} \sum_{k=-N}^{N} ka_k, \]  \hspace{1cm} (1.19)

where \( a = R/2 \) is a half length of a centre crack in an infinite plane. This value is obtained from (1.8) by noting that \( z = a \) corresponds to \( \zeta = 1 \).
Chapter 2

Experiment
2.1 Image correlation (IC)

A patch (interrogation window) from one image is mapped onto a second image such that the correlation between the two patches is maximised. Various correlation functions can be used (Sutton et al., 1999), e.g.:

\[
C(\Delta x, \Delta y) = \sum_{x=0, y=0}^{x<n, y<n} I_1(x, y) \times I_2(x+\Delta x, y+\Delta y) \tag{2.1}
\]

where

\[-\frac{n}{2} < \Delta x; \Delta y < \frac{n}{2}, \tag{2.2}\]

and \(I_1, I_2\) - the intensities of the two images, \(\Delta x\) and \(\Delta y\) - the horizontal and vertical displacements respectively and \(n\) - the window size.

In this experiment DaVis 6.2 image correlation software made by LaVision was used.
2.2 Specimen

To promote mixed mode $K_1 + K_{II}$ stress state, the centre-cracked specimen proposed by Otsuka et al. (1987) was used. The gauge section width and thickness are 20 mm and 3.1 mm respectively. The notch length was $2a = 40$ mm. There was no crack. Material was Al 7010 alloy.

Figure 2.1: Centre-cracked specimen, all dimensions are in mm
2.3 Surface preparation

Successful application of the IC technique requires high contrast images with randomly distributed intensity. To achieve this the specimen surface was scratched with 800, 600 and 400 SiC grit paper. A net of scratches of varying depth, width and orientation was generated.

The specimen was illuminated by a ring light.

The image has high contrast and many fine details. Very few pixels are saturated (green colour).

Figure 2.2: An image of the specimen surface showing a net of scratches. The notch is on the left.
2.4 Experimental setup

Figure 2.3: Experimental setup showing the specimen loaded in mixed mode via two pairs of thick steel plates, and the optical system.
2.5 Mode I

2.5.1 Horizontal displacements, IC results

Displacement resolution is 3.7 μm/pixel. Noise in displacement data is greatly reduced compared to the 2004 results.

Figure 2.4: Horizontal displacement field, in pixels.

Figure 2.5: For noise comparison only. Old horizontal displacement data for mode I, in μm. The image is taken from A. Shterenlikht et al (2004) presentations at ASTM and BSSM meetings.
2.5.2 Vertical displacements, IC results

Displacement resolution is 3.7 $\mu$m/pixel. Noise in displacement data is greatly reduced compared to the 2004 results.

Figure 2.6: Vertical displacement field, in pixels.

Figure 2.7: For noise comparison only. Old vertical displacement data for mode I, in $\mu$m. The image is taken from A. Shterenlikht et al (2004) presentations at ASTM and BSSM meetings.
2.5.3 Crack tip location

Fig. 2.8 shows that the crack tip (notch tip) location as indicated by the shape of the horizontal and the vertical displacement fields is far from the notch tip location in the initial or the final images.

So the crack tip location must be determined from the displacement data. This, however, can only be done approximately.

![Figure 2.8](image_url)

Figure 2.8: Fragments of four images aligned vertically, showing a) image of the undeformed sample, b) image of the sample surface at the end of the deformation, c) vertical displacement field, and d) horizontal displacement field.
2.5.4 Datapoint selection

Only some displacement values (data points) are used for the solution of (1.15)–(1.16). These are taken from 6 concentric circles centered at the crack tip from a sector [-100 deg ; 100 deg], 30 points on each circle, 180 points in total. The minimum and the maximum circle radii are 1 mm and 1.4 mm respectively.

Figure 2.9: Displacement data points used for the solution.
2.5.5 Quality assessment of the experimental data

There is only one outlier point in the data array chosen for the solution. This point can be excluded manually or with the help of some filtering algorithm. In this example this outlier point was kept.

Figure 2.10: Displacement data points used for the solution.
2.5.6 Calculated SIFs

The nominal loading was under pure mode I with $K_I = 21$ MPa$\sqrt{m}$.

The crack tip location was taken where the visible notch tip was.

The crack length, $a$, was taken at first equal to the notch length, $a = 20$ mm.

The 121 term series expansion was used.

<table>
<thead>
<tr>
<th>$a$, mm</th>
<th>$K_I$, MPa$\sqrt{m}$</th>
<th>$K_{II}$, MPa$\sqrt{m}$</th>
<th>$K_I$ error, %</th>
</tr>
</thead>
<tbody>
<tr>
<td>20.00</td>
<td>161</td>
<td>negative</td>
<td>667</td>
</tr>
<tr>
<td>18.80</td>
<td>31</td>
<td>2</td>
<td>48</td>
</tr>
<tr>
<td>18.75</td>
<td>27</td>
<td>7</td>
<td>29</td>
</tr>
<tr>
<td>18.70</td>
<td>18</td>
<td>5</td>
<td>14</td>
</tr>
</tbody>
</table>

Table 2.1: Calculated SIFs compared with the nominal value for pure mode I loading.
Chapter 3

Conclusions

1. The displacement algorithm proved successful.

2. The image quality is of major importance for the performance of the IC technique.

3. Some mode II component is probably caused by specimen mis-alignment during the experiment.

4. The calculated $K_1$ value is extremely sensitive to the crack length and/or the crack tip location with respect to the measured displacement field.

5. Next step: crack tip location is found as a part of the solution. That means solving a large system of nonlinear equations. GA technique used by Díaz Garrido (2004) is a possible candidate.
Bibliography


