

Calculation of mixed mode (I + II) SIFs using
in-plane displacement fields near crack

Anton Shterenlikht

Pablo López-Crespo

Materials Science Centre
The University of Manchester
Grosvenor Street
Manchester M1 7HS
UK

Mechanical Engineering Department
The University of Sheffield
Mappin Street
Sheffield S1 3JD
UK

circa 2004-2005

Contents

1	Introduction	5
1.1	Fundamental equations	5
1.2	Conformal mapping	6
1.3	Fourier series representation	8
1.4	Truncated Fourier series	14
2	Mapping a unit circle onto an ellipse	17
3	Stress Intensity Factors	23
4	Westergaard crack tip fields	25
5	Verification	27
5.1	Linear system with unique solution	27
5.2	Overdetermined system	29
5.2.1	Crack length	29
6	Genetic Algorithm	37
	Bibliography	38

Chapter 1

Introduction

1.1 Fundamental equations

Let's assume a body lying in complex plane Z . The coordinate of any point in this plane can be expressed through a complex variable $z = x_1 + ix_2$. Displacements and stresses can be expressed through two analytical functions

$$\phi = \phi(z) \quad (1.1)$$

$$\psi = \psi(z) \quad (1.2)$$

as follows (Muskhelishvili, 1977):

$$2\mu(u_1 + iu_2) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} \quad (1.3)$$

$$\sigma_{11} + i\sigma_{12} = \phi'(z) + \overline{\phi'(z)} - z\overline{\phi''(z)} - \overline{\psi'(z)} \quad (1.4)$$

$$\sigma_{22} - i\sigma_{12} = \phi'(z) + \overline{\phi'(z)} + z\overline{\phi''(z)} + \overline{\psi'(z)} \quad (1.5)$$

where

$$\mu = \frac{E}{2(1+\nu)}; \quad \kappa = 3 - 4\nu \text{ (plane strain); } \quad \kappa = \frac{3 - \nu}{1 + \nu} \text{ (plane stress)} \quad (1.6)$$

u_i are displacements and σ_{ij} are stresses, $i, j = 1, 2$. The overbar denotes complex conjugate.

In the following we shall look at the class of problems which have one common feature – a traction-free contour, L , resembling a crack.

The boundary condition can be expressed as

$$\phi(t) + \overline{t\phi'(t)} + \overline{\psi(t)} = i \int_s (\sigma_{1n} + i\sigma_{2n}) ds \quad (1.7)$$

where t is a point on L , s is the arc on L , and σ_{1n} and σ_{2n} are stress components acting on a plane perpendicular to normal \vec{n} at point t .

For the traction-free contour L , $\sigma_{1n} = \sigma_{2n} = 0$ and the boundary condition can simply be written as

$$\phi(t) + t\overline{\phi'(t)} + \overline{\psi(t)} = 0. \quad (1.8)$$

The body under analysis can be simply or multiply connected, finite or infinite.

1.2 Conformal mapping

The boundary conditions on L , that the normal and the shear stresses are zero, can sometimes be expressed more easily in another plane S which has a complex variable ζ related to z through a complex valued function ω :

$$z = \omega(\zeta). \quad (1.9)$$

It can be shown that under very general conditions expression (1.9) represents a conformal mapping of S onto Z . Plane S is called the ‘mapping’ plane.

From (1.1), (1.2) and (1.9) one can obtain

$$d\phi = \frac{d\phi}{dz}dz = \frac{d\phi}{dz} \frac{d\omega}{d\zeta}d\zeta \quad (1.10)$$

or

$$\frac{d\phi}{d\zeta} = \frac{d\phi}{dz} \frac{d\omega}{d\zeta} \quad (1.11)$$

or

$$\frac{d\phi}{dz} = \frac{d\phi/d\zeta}{d\omega/d\zeta} \quad (1.12)$$

or finally

$$\phi'(z) = \frac{\phi'(\zeta)}{\omega'(\zeta)}. \quad (1.13)$$

Similarly to (1.12)

$$\frac{d\phi'(z)}{dz} = \frac{d\phi'(z)/d\zeta}{d\omega/d\zeta} \quad (1.14)$$

or using (1.13)

$$\frac{d\phi'(z)}{dz} = \frac{\frac{d}{d\zeta}(\phi'(\zeta)/\omega'(\zeta))}{d\omega/d\zeta} \quad (1.15)$$

or finally

$$\phi''(z) = \frac{\phi''(\zeta)\omega'(\zeta) - \phi'(\zeta)\omega''(\zeta)}{[\omega'(\zeta)]^3}. \quad (1.16)$$

It is easy to prove, using the fundamental properties of functions of complex variable, that the complex conjugates of (1.13) and (1.16) will be

$$\overline{\phi'(z)} = \frac{\overline{\phi'(\zeta)}}{\overline{\omega'(\zeta)}} \quad (1.17)$$

and

$$\overline{\phi''(z)} = \frac{\overline{\phi''(\zeta)} \overline{\omega'(\zeta)} - \overline{\phi'(\zeta)} \overline{\omega''(\zeta)}}{[\overline{\omega'(\zeta)}]^3}. \quad (1.18)$$

Similarly to (1.17)

$$\overline{\psi'(z)} = \frac{\overline{\psi'(\zeta)}}{\overline{\omega'(\zeta)}} \quad (1.19)$$

Substituting (1.9), (1.13), (1.17)-(1.19) into (1.3)-(1.5) one obtains the basic expressions in the mapping plane:

$$2\mu(u_1 + iu_2) = \varkappa\phi(\zeta) - \frac{\omega(\zeta)}{\omega'(\zeta)}\overline{\phi'(\zeta)} - \overline{\psi(\zeta)} \quad (1.20)$$

$$\sigma_{11} + i\sigma_{12} = \frac{\phi'(\zeta)}{\omega'(\zeta)} + \frac{\overline{\phi'(\zeta)}}{\overline{\omega'(\zeta)}} - \omega(\zeta) \frac{\overline{\phi''(\zeta)} \overline{\omega'(\zeta)} - \overline{\phi'(\zeta)} \overline{\omega''(\zeta)}}{[\overline{\omega'(\zeta)}]^3} - \frac{\overline{\psi'(\zeta)}}{\overline{\omega'(\zeta)}} \quad (1.21)$$

$$\sigma_{22} - i\sigma_{12} = \frac{\phi'(\zeta)}{\omega'(\zeta)} + \frac{\overline{\phi'(\zeta)}}{\overline{\omega'(\zeta)}} + \omega(\zeta) \frac{\overline{\phi''(\zeta)} \overline{\omega'(\zeta)} - \overline{\phi'(\zeta)} \overline{\omega''(\zeta)}}{[\overline{\omega'(\zeta)}]^3} + \frac{\overline{\psi'(\zeta)}}{\overline{\omega'(\zeta)}} \quad (1.22)$$

Function ω is chosen so that contour L of plane Z maps onto a unit circle γ in the mapping plane. Accordingly it is easier to use polar coordinates (r, θ) in plane S :

$$\zeta = re^{i\theta} \quad (1.23)$$

The points on γ are denoted

$$\eta = e^{i\theta}, \quad (1.24)$$

so that $\eta \equiv \zeta|_{r=1}$.

Substituting (1.9) and (1.24) into (1.8) one obtains

$$\phi(\eta) + \frac{\omega(\eta)}{\omega'(\eta)}\overline{\phi'(\eta)} + \overline{\psi(\eta)} = 0 \quad (1.25)$$

1.3 Fourier series representation

The analytical functions can be represented as infinite Fourier series:

$$\phi(\zeta) = \sum_{-\infty}^{+\infty} a_k \zeta^k; \quad \text{and} \quad \phi'(\zeta) = \sum_{-\infty}^{+\infty} k a_k \zeta^{k-1} \quad (1.26)$$

$$\phi''(\zeta) = \sum_{-\infty}^{+\infty} k(k-1) a_k \zeta^{k-2} \quad (1.27)$$

$$\psi(\zeta) = \sum_{-\infty}^{+\infty} b_k \zeta^k; \quad \psi'(\zeta) = \sum_{-\infty}^{+\infty} k b_k \zeta^{k-1} \quad (1.28)$$

where a_k and b_k are complex coefficients.

If the contour L is smooth than $\omega(\eta)/\overline{\omega'(\eta)}$ can also be represented by its Fourier series:

$$\frac{\omega(\eta)}{\overline{\omega'(\eta)}} = \sum_{-\infty}^{+\infty} c_k \eta^k \quad (1.29)$$

where c_k are complex coefficients.

Unless said otherwise in the following the summation is always assumed from minus to plus infinity. Accordingly only summation index is given.

By substituting (1.26), (1.28) and (1.29) into (1.25) and taking into account that

$$\bar{\eta} = e^{-i\theta} = \eta^{-1} \quad (1.30)$$

one gets:

$$\sum_k a_k \eta^k + \sum_l c_l \eta^l \sum_k k \bar{a}_k \eta^{-k+1} + \sum_k \bar{b}_k \eta^{-k} = 0 \quad (1.31)$$

or multiplying the series in the middle of the left-hand side

$$\sum_k a_k \eta^k + \sum_l \sum_k k \bar{a}_k c_l \eta^{l-k+1} + \sum_k \bar{b}_k \eta^{-k} = 0 \quad (1.32)$$

Comparing coefficients of η^m ($m = l - k + 1$ for the second term and $m = -k$ for the third term) one gets:

$$a_m + \sum_k k \bar{a}_k c_{m+k-1} + \bar{b}_{-m} = 0 \quad (1.33)$$

or

$$b_k = -\bar{a}_{-k} - \sum_l l a_l \bar{c}_{l-k-1} \quad (1.34)$$

With that $\psi(\zeta)$ can be rewritten as:

$$\psi(\zeta) = \sum_k \left(-\bar{a}_{-k} - \sum_l l a_l \bar{c}_{l-k-1} \right) \zeta^k \quad (1.35)$$

Substituting (1.26) and (1.35) into (1.20) one obtains

$$2\mu(u_1 + iu_2) = \varkappa \sum_k a_k \zeta^k - \frac{\omega(\zeta)}{\omega'(\zeta)} \sum_k k \bar{a}_k \zeta^{\overline{k-1}} - \sum_k \left(-a_{-k} - \sum_l l \bar{a}_l c_{l-k-1} \right) \bar{\zeta}^k \quad (1.36)$$

Equation (1.36) is a linear equation with respect to infinite number of complex unknowns, a_k . If one introduce

$$a_k = \alpha_k + i\beta_k \quad (1.37)$$

then (1.36) will split into two real linear equations with respect to infinite number of real unknowns, $\alpha_k = \text{Re}a_k$ and $\beta_k = \text{Im}a_k$.

Let's obtain the coefficients of the two real equations.

Let's introduce

$$\Omega(\zeta) = \frac{\omega(\zeta)}{\omega'(\zeta)}. \quad (1.38)$$

Because the series in (1.36) are from minus to plus infinity, summation index k may be substituted by $-k$. Applying this property to the last series in (1.36) and using (1.38), eqn. (1.36) can be rewritten:

$$\varkappa \sum_k a_k \zeta^k - \Omega \sum_k k \bar{a}_k \zeta^{\overline{k-1}} + \sum_k \left(a_k + \sum_l l \bar{a}_l c_{l+k-1} \right) \bar{\zeta}^{-k} = 2\mu(u_1 + iu_2) \quad (1.39)$$

or using (1.37)

$$\begin{aligned} & \varkappa \sum_k (\alpha_k + i\beta_k) (\text{Re}\zeta^k + i\text{Im}\zeta^k) - \\ & -\Omega \sum_k k (\alpha_k - i\beta_k) (\text{Re}\zeta^{\overline{k-1}} - i\text{Im}\zeta^{\overline{k-1}}) + \\ & + \sum_k \left((\alpha_k + i\beta_k) + \sum_l l (\alpha_l - i\beta_l) (\text{Re}c_{l+k-1} + i\text{Im}c_{l+k-1}) \right) \times \\ & \times (\text{Re}\zeta^{-k} - i\text{Im}\zeta^{-k}) = 2\mu(u_1 + iu_2) \quad (1.40) \end{aligned}$$

or multiplying the terms under summation signs

$$\begin{aligned}
& \varkappa \sum_k \alpha_k \operatorname{Re} \zeta^k + i \varkappa \sum_k \alpha_k \operatorname{Im} \zeta^k + i \varkappa \sum_k \beta_k \operatorname{Re} \zeta^k - \varkappa \sum_k \beta_k \operatorname{Im} \zeta^k - \\
& \quad - (\operatorname{Re} \Omega + i \operatorname{Im} \Omega) \left(\sum_k k \alpha_k \operatorname{Re} \zeta^{k-1} - i \sum_k k \alpha_k \operatorname{Im} \zeta^{k-1} \right. \\
& \quad \quad \left. - i \sum_k k \beta_k \operatorname{Re} \zeta^{k-1} - \sum_k k \beta_k \operatorname{Im} \zeta^{k-1} \right) + \\
& \quad \quad + \sum_k (\alpha_k + i \beta_k) (\operatorname{Re} \zeta^{-k} - i \operatorname{Im} \zeta^{-k}) + \\
& + \sum_k \sum_l l (\alpha_l - i \beta_l) (\operatorname{Re} c_{l+k-1} + i \operatorname{Im} c_{l+k-1}) (\operatorname{Re} \zeta^{-k} - i \operatorname{Im} \zeta^{-k}) = \\
& \qquad \qquad \qquad = 2\mu(u_1 + iu_2) \quad (1.41)
\end{aligned}$$

or multiplying further

$$\begin{aligned}
& \varkappa \sum_k \alpha_k \operatorname{Re} \zeta^k + i \varkappa \sum_k \alpha_k \operatorname{Im} \zeta^k + i \varkappa \sum_k \beta_k \operatorname{Re} \zeta^k - \varkappa \sum_k \beta_k \operatorname{Im} \zeta^k - \\
& \quad - \left(\operatorname{Re} \Omega \sum_k k \alpha_k \operatorname{Re} \zeta^{k-1} - i \operatorname{Re} \Omega \sum_k k \alpha_k \operatorname{Im} \zeta^{k-1} - \right. \\
& \quad \quad - i \operatorname{Re} \Omega \sum_k k \beta_k \operatorname{Re} \zeta^{k-1} - \operatorname{Re} \Omega \sum_k k \beta_k \operatorname{Im} \zeta^{k-1} + \\
& \quad \quad + i \operatorname{Im} \Omega \sum_k k \alpha_k \operatorname{Re} \zeta^{k-1} + \operatorname{Im} \Omega \sum_k k \alpha_k \operatorname{Im} \zeta^{k-1} + \\
& \quad \quad \left. + \operatorname{Im} \Omega \sum_k k \beta_k \operatorname{Re} \zeta^{k-1} - i \operatorname{Im} \Omega \sum_k k \beta_k \operatorname{Im} \zeta^{k-1} \right) + \\
& + \sum_k \alpha_k \operatorname{Re} \zeta^{-k} - i \sum_k \alpha_k \operatorname{Im} \zeta^{-k} + i \sum_k \beta_k \operatorname{Re} \zeta^{-k} + \sum_k \beta_k \operatorname{Im} \zeta^{-k} + \\
& \quad + \sum_k \sum_l l (\alpha_l - i \beta_l) (\operatorname{Re} c_{l+k-1} \operatorname{Re} \zeta^{-k} - i \operatorname{Re} c_{l+k-1} \operatorname{Im} \zeta^{-k} + \\
& \quad \quad + i \operatorname{Im} c_{l+k-1} \operatorname{Re} \zeta^{-k} + \operatorname{Im} c_{l+k-1} \operatorname{Im} \zeta^{-k}) = 2\mu(u_1 + iu_2) \quad (1.42)
\end{aligned}$$

or multiplying further

$$\begin{aligned}
& \varkappa \sum_k \alpha_k \operatorname{Re} \zeta^k + i \varkappa \sum_k \alpha_k \operatorname{Im} \zeta^k + i \varkappa \sum_k \beta_k \operatorname{Re} \zeta^k - \varkappa \sum_k \beta_k \operatorname{Im} \zeta^k - \\
& \quad - \operatorname{Re} \Omega \sum_k k \alpha_k \operatorname{Re} \zeta^{k-1} + i \operatorname{Re} \Omega \sum_k k \alpha_k \operatorname{Im} \zeta^{k-1} + \\
& \quad + i \operatorname{Re} \Omega \sum_k k \beta_k \operatorname{Re} \zeta^{k-1} + \operatorname{Re} \Omega \sum_k k \beta_k \operatorname{Im} \zeta^{k-1} - \\
& \quad - i \operatorname{Im} \Omega \sum_k k \alpha_k \operatorname{Re} \zeta^{k-1} - \operatorname{Im} \Omega \sum_k k \alpha_k \operatorname{Im} \zeta^{k-1} - \\
& \quad - \operatorname{Im} \Omega \sum_k k \beta_k \operatorname{Re} \zeta^{k-1} + i \operatorname{Im} \Omega \sum_k k \beta_k \operatorname{Im} \zeta^{k-1} + \\
& + \sum_k \alpha_k \operatorname{Re} \zeta^{-k} - i \sum_k \alpha_k \operatorname{Im} \zeta^{-k} + i \sum_k \beta_k \operatorname{Re} \zeta^{-k} + \sum_k \beta_k \operatorname{Im} \zeta^{-k} + \\
& \quad + \sum_k \sum_l l \alpha_l \operatorname{Re} c_{l+k-1} \operatorname{Re} \zeta^{-k} - i \sum_k \sum_l l \alpha_l \operatorname{Re} c_{l+k-1} \operatorname{Im} \zeta^{-k} + \\
& \quad + i \sum_k \sum_l l \alpha_l \operatorname{Im} c_{l+k-1} \operatorname{Re} \zeta^{-k} + \sum_k \sum_l l \alpha_l \operatorname{Im} c_{l+k-1} \operatorname{Im} \zeta^{-k} - \\
& \quad - i \sum_k \sum_l l \beta_l \operatorname{Re} c_{l+k-1} \operatorname{Re} \zeta^{-k} - \sum_k \sum_l l \beta_l \operatorname{Re} c_{l+k-1} \operatorname{Im} \zeta^{-k} + \\
& \quad + \sum_k \sum_l l \beta_l \operatorname{Im} c_{l+k-1} \operatorname{Re} \zeta^{-k} - i \sum_k \sum_l l \beta_l \operatorname{Im} c_{l+k-1} \operatorname{Im} \zeta^{-k} \\
& \hspace{15em} = 2\mu(u_1 + iu_2) \quad (1.43)
\end{aligned}$$

or swapping k and l in the double sums (this operation is permissible as both

k and l run from minus to plus infinity)

$$\begin{aligned}
& \varkappa \sum_k \alpha_k \operatorname{Re} \zeta^k + i \varkappa \sum_k \alpha_k \operatorname{Im} \zeta^k + i \varkappa \sum_k \beta_k \operatorname{Re} \zeta^k - \varkappa \sum_k \beta_k \operatorname{Im} \zeta^k - \\
& \quad - \operatorname{Re} \Omega \sum_k k \alpha_k \operatorname{Re} \zeta^{k-1} + i \operatorname{Re} \Omega \sum_k k \alpha_k \operatorname{Im} \zeta^{k-1} + \\
& \quad + i \operatorname{Re} \Omega \sum_k k \beta_k \operatorname{Re} \zeta^{k-1} + \operatorname{Re} \Omega \sum_k k \beta_k \operatorname{Im} \zeta^{k-1} - \\
& \quad - i \operatorname{Im} \Omega \sum_k k \alpha_k \operatorname{Re} \zeta^{k-1} - \operatorname{Im} \Omega \sum_k k \alpha_k \operatorname{Im} \zeta^{k-1} - \\
& \quad - \operatorname{Im} \Omega \sum_k k \beta_k \operatorname{Re} \zeta^{k-1} + i \operatorname{Im} \Omega \sum_k k \beta_k \operatorname{Im} \zeta^{k-1} + \\
& + \sum_k \alpha_k \operatorname{Re} \zeta^{-k} - i \sum_k \alpha_k \operatorname{Im} \zeta^{-k} + i \sum_k \beta_k \operatorname{Re} \zeta^{-k} + \sum_k \beta_k \operatorname{Im} \zeta^{-k} + \\
& \quad + \sum_k k \alpha_k \sum_l \operatorname{Re} c_{l+k-1} \operatorname{Re} \zeta^{-l} - i \sum_k k \alpha_k \sum_l \operatorname{Re} c_{l+k-1} \operatorname{Im} \zeta^{-l} + \\
& \quad + i \sum_k k \alpha_k \sum_l \operatorname{Im} c_{l+k-1} \operatorname{Re} \zeta^{-l} + \sum_k k \alpha_k \sum_l \operatorname{Im} c_{l+k-1} \operatorname{Im} \zeta^{-l} - \\
& \quad - i \sum_k k \beta_k \sum_l \operatorname{Re} c_{l+k-1} \operatorname{Re} \zeta^{-l} - \sum_k k \beta_k \sum_l \operatorname{Re} c_{l+k-1} \operatorname{Im} \zeta^{-l} + \\
& \quad + \sum_k k \beta_k \sum_l \operatorname{Im} c_{l+k-1} \operatorname{Re} \zeta^{-l} - i \sum_k k \beta_k \sum_l \operatorname{Im} c_{l+k-1} \operatorname{Im} \zeta^{-l} \\
& \hspace{20em} = 2\mu(u_1 + iu_2) \quad (1.44)
\end{aligned}$$

Equation (1.44) can be written as a system of two real equations:

$$\begin{aligned}
& \varkappa \sum_k \alpha_k \operatorname{Re} \zeta^k - \varkappa \sum_k \beta_k \operatorname{Im} \zeta^k - \operatorname{Re} \Omega \sum_k k \alpha_k \operatorname{Re} \zeta^{k-1} + \\
& + \operatorname{Re} \Omega \sum_k k \beta_k \operatorname{Im} \zeta^{k-1} - \operatorname{Im} \Omega \sum_k k \alpha_k \operatorname{Im} \zeta^{k-1} - \operatorname{Im} \Omega \sum_k k \beta_k \operatorname{Re} \zeta^{k-1} + \\
& + \sum_k \alpha_k \operatorname{Re} \zeta^{-k} + \sum_k \beta_k \operatorname{Im} \zeta^{-k} + \sum_k k \alpha_k \sum_l \operatorname{Re} c_{l+k-1} \operatorname{Re} \zeta^{-l} + \\
& + \sum_k k \alpha_k \sum_l \operatorname{Im} c_{l+k-1} \operatorname{Im} \zeta^{-l} - \sum_k k \beta_k \sum_l \operatorname{Re} c_{l+k-1} \operatorname{Im} \zeta^{-l} + \\
& \hspace{15em} + \sum_k k \beta_k \sum_l \operatorname{Im} c_{l+k-1} \operatorname{Re} \zeta^{-l} = 2\mu u_1 \quad (1.45)
\end{aligned}$$

$$\begin{aligned}
& i\mathcal{Z} \sum_k \alpha_k \operatorname{Im} \zeta^k + i\mathcal{Z} \sum_k \beta_k \operatorname{Re} \zeta^k + i\operatorname{Re} \Omega \sum_k k \alpha_k \operatorname{Im} \zeta^{k-1} + \\
& + i\operatorname{Re} \Omega \sum_k k \beta_k \operatorname{Re} \zeta^{k-1} - i\operatorname{Im} \Omega \sum_k k \alpha_k \operatorname{Re} \zeta^{k-1} + i\operatorname{Im} \Omega \sum_k k \beta_k \operatorname{Im} \zeta^{k-1} - \\
& - i \sum_k \alpha_k \operatorname{Im} \zeta^{-k} + i \sum_k \beta_k \operatorname{Re} \zeta^{-k} - i \sum_k k \alpha_k \sum_l \operatorname{Re} c_{l+k-1} \operatorname{Im} \zeta^{-l} + \\
& + i \sum_k k \alpha_k \sum_l \operatorname{Im} c_{l+k-1} \operatorname{Re} \zeta^{-l} - i \sum_k k \beta_k \sum_l \operatorname{Re} c_{l+k-1} \operatorname{Re} \zeta^{-l} - \\
& - i \sum_k k \beta_k \sum_l \operatorname{Im} c_{l+k-1} \operatorname{Im} \zeta^{-l} = i2\mu u_2
\end{aligned} \tag{1.46}$$

or

$$\begin{aligned}
& \sum_k \{ \mathcal{Z} \operatorname{Re} \zeta^k - k \operatorname{Re} \Omega \operatorname{Re} \zeta^{k-1} - k \operatorname{Im} \Omega \operatorname{Im} \zeta^{k-1} + \operatorname{Re} \zeta^{-k} + \\
& + k \sum_l \operatorname{Re} c_{l+k-1} \operatorname{Re} \zeta^{-l} + k \sum_l \operatorname{Im} c_{l+k-1} \operatorname{Im} \zeta^{-l} \} \alpha_k + \\
& + \sum_k \{ -\mathcal{Z} \operatorname{Im} \zeta^k + k \operatorname{Re} \Omega \operatorname{Im} \zeta^{k-1} - k \operatorname{Im} \Omega \operatorname{Re} \zeta^{k-1} + \operatorname{Im} \zeta^{-k} - \\
& - k \sum_l \operatorname{Re} c_{l+k-1} \operatorname{Im} \zeta^{-l} + k \sum_l \operatorname{Im} c_{l+k-1} \operatorname{Re} \zeta^{-l} \} \beta_k = 2\mu u_1
\end{aligned} \tag{1.47}$$

$$\begin{aligned}
& \sum_k \{ \mathcal{Z} \operatorname{Im} \zeta^k + k \operatorname{Re} \Omega \operatorname{Im} \zeta^{k-1} - k \operatorname{Im} \Omega \operatorname{Re} \zeta^{k-1} - \operatorname{Im} \zeta^{-k} - \\
& - k \sum_l \operatorname{Re} c_{l+k-1} \operatorname{Im} \zeta^{-l} + k \sum_l \operatorname{Im} c_{l+k-1} \operatorname{Re} \zeta^{-l} \} \alpha_k + \\
& + \sum_k \{ \mathcal{Z} \operatorname{Re} \zeta^k + k \operatorname{Re} \Omega \operatorname{Re} \zeta^{k-1} + k \operatorname{Im} \Omega \operatorname{Im} \zeta^{k-1} + \operatorname{Re} \zeta^{-k} - \\
& - k \sum_l \operatorname{Re} c_{l+k-1} \operatorname{Re} \zeta^{-l} - k \sum_l \operatorname{Im} c_{l+k-1} \operatorname{Im} \zeta^{-l} \} \beta_k = 2\mu u_2
\end{aligned} \tag{1.48}$$

or finally

$$\sum_k A_k \alpha_k + \sum_k B_k \beta_k = 2\mu u_1 \tag{1.49}$$

$$\sum_k C_k \alpha_k + \sum_k D_k \beta_k = 2\mu u_2 \tag{1.50}$$

where

$$A_k = \varkappa \operatorname{Re} \zeta^k - k \operatorname{Re} \Omega \operatorname{Re} \zeta^{k-1} - k \operatorname{Im} \Omega \operatorname{Im} \zeta^{k-1} + \operatorname{Re} \zeta^{-k} + k \sum_l \operatorname{Re} c_{l+k-1} \operatorname{Re} \zeta^{-l} + k \sum_l \operatorname{Im} c_{l+k-1} \operatorname{Im} \zeta^{-l} \quad (1.51)$$

$$B_k = -\varkappa \operatorname{Im} \zeta^k + k \operatorname{Re} \Omega \operatorname{Im} \zeta^{k-1} - k \operatorname{Im} \Omega \operatorname{Re} \zeta^{k-1} + \operatorname{Im} \zeta^{-k} - k \sum_l \operatorname{Re} c_{l+k-1} \operatorname{Im} \zeta^{-l} + k \sum_l \operatorname{Im} c_{l+k-1} \operatorname{Re} \zeta^{-l} \quad (1.52)$$

$$C_k = \varkappa \operatorname{Im} \zeta^k + k \operatorname{Re} \Omega \operatorname{Im} \zeta^{k-1} - k \operatorname{Im} \Omega \operatorname{Re} \zeta^{k-1} - \operatorname{Im} \zeta^{-k} - k \sum_l \operatorname{Re} c_{l+k-1} \operatorname{Im} \zeta^{-l} + k \sum_l \operatorname{Im} c_{l+k-1} \operatorname{Re} \zeta^{-l} \quad (1.53)$$

$$D_k = \varkappa \operatorname{Re} \zeta^k + k \operatorname{Re} \Omega \operatorname{Re} \zeta^{k-1} + k \operatorname{Im} \Omega \operatorname{Im} \zeta^{k-1} + \operatorname{Re} \zeta^{-k} - k \sum_l \operatorname{Re} c_{l+k-1} \operatorname{Re} \zeta^{-l} - k \sum_l \operatorname{Im} c_{l+k-1} \operatorname{Im} \zeta^{-l} \quad (1.54)$$

1.4 Truncated Fourier series

To get a limited number of unknowns one has to introduce limits for series in (1.26)-(1.28).

Let's designate N as a limit for the series representation of the two analytical functions:

$$\phi(\zeta) = \sum_{k=-N}^N a_k \zeta^k; \quad \psi(\zeta) = \sum_{k=-N}^N b_k \zeta^k. \quad (1.55)$$

where from (1.34)

$$b_k = -\bar{a}_{-k} - \sum_{l=-N}^{l=N} l a_l \bar{c}_{l-k-1}. \quad (1.56)$$

Let's also limit the series in (1.51)-(1.54) at M . Then (1.49)-(1.54) will be written as:

$$\sum_{k=-N}^N A_k \alpha_k + \sum_{k=-N}^N B_k \beta_k = 2\mu u_1 \quad (1.57)$$

$$\sum_{k=-N}^N C_k \alpha_k + \sum_{k=-N}^N D_k \beta_k = 2\mu u_2 \quad (1.58)$$

where

$$\begin{aligned} A_k = & \varkappa \operatorname{Re} \zeta^k - k \operatorname{Re} \Omega \operatorname{Re} \zeta^{k-1} - k \operatorname{Im} \Omega \operatorname{Im} \zeta^{k-1} + \operatorname{Re} \zeta^{-k} + \\ & + k \sum_{l=-M}^M \operatorname{Re} c_{l+k-1} \operatorname{Re} \zeta^{-l} + k \sum_{l=-M}^M \operatorname{Im} c_{l+k-1} \operatorname{Im} \zeta^{-l} \end{aligned} \quad (1.59)$$

$$\begin{aligned} B_k = & -\varkappa \operatorname{Im} \zeta^k + k \operatorname{Re} \Omega \operatorname{Im} \zeta^{k-1} - k \operatorname{Im} \Omega \operatorname{Re} \zeta^{k-1} + \operatorname{Im} \zeta^{-k} - \\ & - k \sum_{l=-M}^M \operatorname{Re} c_{l+k-1} \operatorname{Im} \zeta^{-l} + k \sum_{l=-M}^M \operatorname{Im} c_{l+k-1} \operatorname{Re} \zeta^{-l} \end{aligned} \quad (1.60)$$

$$\begin{aligned} C_k = & \varkappa \operatorname{Im} \zeta^k + k \operatorname{Re} \Omega \operatorname{Im} \zeta^{k-1} - k \operatorname{Im} \Omega \operatorname{Re} \zeta^{k-1} - \operatorname{Im} \zeta^{-k} - \\ & - k \sum_{l=-M}^M \operatorname{Re} c_{l+k-1} \operatorname{Im} \zeta^{-l} + k \sum_{l=-M}^M \operatorname{Im} c_{l+k-1} \operatorname{Re} \zeta^{-l} \end{aligned} \quad (1.61)$$

$$\begin{aligned} D_k = & \varkappa \operatorname{Re} \zeta^k + k \operatorname{Re} \Omega \operatorname{Re} \zeta^{k-1} + k \operatorname{Im} \Omega \operatorname{Im} \zeta^{k-1} + \operatorname{Re} \zeta^{-k} - \\ & - k \sum_{l=-M}^M \operatorname{Re} c_{l+k-1} \operatorname{Re} \zeta^{-l} - k \sum_{l=-M}^M \operatorname{Im} c_{l+k-1} \operatorname{Im} \zeta^{-l} \end{aligned} \quad (1.62)$$

The system (1.57)-(1.58) has $2N + 1$ unknowns α_k and $2N + 1$ unknowns β_k , or $2(2N + 1)$ unknowns in total and 2 equations. This system is written for only one point ζ . If one writes this system for p points ζ then the system will have $2p$ equations:

$$\sum_{k=-N}^N A_k^j \alpha_k + \sum_{k=-N}^N B_k^j \beta_k = 2\mu u_1^j \quad (1.63)$$

$$\sum_{k=-N}^N C_k^j \alpha_k + \sum_{k=-N}^N D_k^j \beta_k = 2\mu u_2^j \quad (1.64)$$

where $j = 1, 2, \dots, p$, u_1^j and u_2^j are displacements at point j , and A_k^j , B_k^j , C_k^j and D_k^j are coefficients defined by (1.59)-(1.62) calculated at point j .

This system can be written in the matrix form as:

$$\mathbf{Ex} = 2\mu \mathbf{u} \quad (1.65)$$

where $\mathbf{E}_{2p \times 2(2N+1)}$ is the matrix of coefficients A_k^j , B_k^j , C_k^j and D_k^j :

$$\mathbf{E} = \begin{pmatrix} A_{-N}^1 & \cdots & A_N^1 & B_{-N}^1 & \cdots & B_N^1 \\ A_{-N}^2 & \cdots & A_N^2 & B_{-N}^2 & \cdots & B_N^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{-N}^p & \cdots & A_N^p & B_{-N}^p & \cdots & B_N^p \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{-N}^1 & \cdots & C_N^1 & D_{-N}^1 & \cdots & D_N^1 \\ C_{-N}^2 & \cdots & C_N^2 & D_{-N}^2 & \cdots & D_N^2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{-N}^p & \cdots & C_N^p & D_{-N}^p & \cdots & D_N^p \end{pmatrix}, \quad (1.66)$$

$\mathbf{x}_{2(2N+1) \times 1}$ is the vector of unknowns:

$$\mathbf{x} = \begin{pmatrix} \alpha_{-N} \\ \cdots \\ \alpha_N \\ \beta_{-N} \\ \cdots \\ \beta_N \end{pmatrix}, \quad (1.67)$$

and $\mathbf{u}_{2p \times 1}$ is the vector of displacements:

$$\mathbf{u} = \begin{pmatrix} u_1^1 \\ \cdots \\ u_1^p \\ u_2^1 \\ \cdots \\ u_2^p \end{pmatrix}. \quad (1.68)$$

If $p = 2N + 1$ then the number of unknowns equals the number of equations and the system will have a unique solution provided $\det \mathbf{E} \neq 0$. The solution can be found by e.g. LU decomposition (Press et al., 1996).

If $p < 2N + 1$ then the number of unknowns is greater than the number of equations and the system is underdetermined, i.e. it will have an infinite number of solutions.

If $p > 2N + 1$ then the number of unknowns is smaller than the number of equations and the system is overdetermined. The solution can be found in the linear least squares sense, e.g. by QR decomposition (Golub and Van Loan, 1996).

Chapter 2

Mapping a unit circle onto an ellipse

The function

$$z = \omega(\zeta) = R \left(\zeta + \frac{m}{\zeta} \right) \quad (2.1)$$

where m is a shape parameter, $0 \leq m \leq 1$, and R is a scale parameter maps a unit circle, $\zeta = e^{i\theta}$, onto an elliptical contour, L , in plane Z . If $m = 0$ then L is a circle of radius R . If $m = 1$ then L is a mathematical crack of zero thickness and length $4R$.

From (2.1)

$$\zeta = \frac{z \pm \sqrt{z^2 - 4R^2m}}{2R}. \quad (2.2)$$

One of the solutions of (2.2) is inside the unit circle, $|\zeta| \leq 1$ and another is outside the unit circle, $|\zeta| \geq 1$. In the following we shall only take $|\zeta| \geq 1$.

From (2.1)

$$\omega'(\zeta) = R \left(1 - \frac{m}{\zeta^2} \right) \quad (2.3)$$

and

$$\overline{\omega'(\zeta)} = R \left(1 - \frac{m}{\bar{\zeta}^2} \right) \quad (2.4)$$

Substituting (2.1) and (2.4) into (1.38) one obtains

$$\Omega = \frac{R(\zeta + m/\zeta)}{R(1 - m/\bar{\zeta}^2)} = \frac{\zeta + m/\zeta}{1 - m/\bar{\zeta}^2}. \quad (2.5)$$

Let's represent Ω by Fourier series as in (1.29)

$$\Omega(\eta) = \frac{\eta + m/\eta}{1 - m/\bar{\eta}^2} \quad (2.6)$$

or using (1.30)

$$\Omega(\eta) = \frac{\eta + m/\eta}{1 - m\eta^2} \quad (2.7)$$

or

$$\begin{aligned} \Omega(\eta) &= \frac{m}{\eta} - \frac{m}{\eta} + \frac{\eta + m/\eta}{1 - m\eta^2} = \frac{m}{\eta} + \frac{-\frac{m}{\eta}(1 - m\eta^2) + \eta + m/\eta}{1 - m\eta^2} \\ &= \frac{m}{\eta} + \frac{m^2\eta + \eta}{1 - m\eta^2} = \frac{m}{\eta} + (m^2 + 1)\eta \cdot \frac{1}{1 - m\eta^2}. \end{aligned} \quad (2.8)$$

The term $1/(1 - m\eta^2)$ can be represented as a series

$$\frac{1}{1 - m\eta^2} = \sum_{k=0}^{+\infty} (m\eta^2)^k \quad (2.9)$$

if $|m\eta^2| < 1$ (Sokolnikoff and Sokolnikoff, 1941).

Because

$$|m\eta^2| = \sqrt{m\eta^2 \cdot m\bar{\eta}^2} = \sqrt{m\eta^2 \cdot m\eta^{-2}} = m \quad (2.10)$$

the series expansion (2.9) is possible only for $m < 1$.

Substituting (2.9) into (2.8) one obtains

$$\Omega(\eta) = \frac{m}{\eta} + (m^2 + 1)\eta \sum_{k=0}^{+\infty} (m\eta^2)^k \quad (2.11)$$

or

$$\Omega(\eta) = m\eta^{-1} + (m^2 + 1) \sum_{k=0}^{+\infty} m^k \eta^{2k+1}. \quad (2.12)$$

Accordingly the c_k coefficients of (1.29) will be expressed as

$$c_k = \begin{cases} m & k = -1 \\ (m^2 + 1)m^{\frac{k-1}{2}} & k = 2n + 1; \quad n = 0, 1, 2, \dots \\ 0 & \text{all other } k \end{cases} \quad (2.13)$$

From (1.29) the truncated Fourier series for $\Omega(\eta)$ will be written as

$$\Omega(\eta) = \sum_{-M}^M c_k \eta^k \quad (2.14)$$

where c_k is defined by (2.13).

The series (2.14) will converge to (2.7) as $M \rightarrow +\infty$. The rate of convergence will be strongly affected by the value of m . The convergence will

slow down as $m \rightarrow 1$. When $m = 1$ the series will diverge (Sokolnikoff and Sokolnikoff, 1941).

On the other hand the contour L will resemble crack less and less as $m \rightarrow 0$.

Accordingly m must be taken big enough, so that L still resembles a crack, but small enough to achieve a satisfactory convergence.

Figures 2.1 and 2.2 show two examples. Figure 2.1 shows the series (2.14) convergence for $m = 0.98$. Satisfactory series convergence can be achieved with $M = 400$.

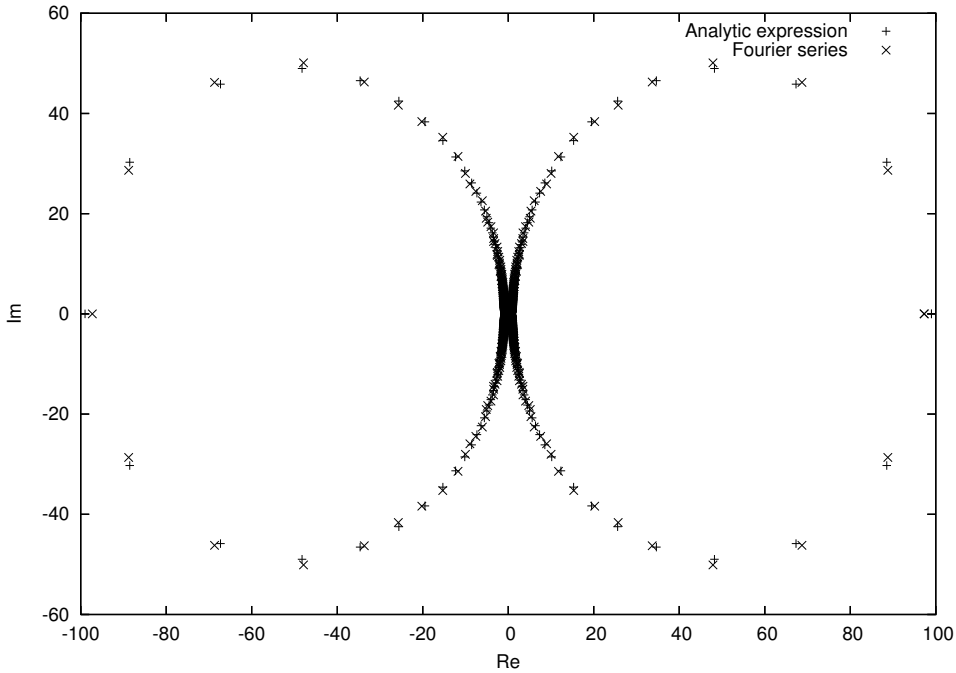


Figure 2.1: Comparison of (2.7) and (2.14) for $m = 0.98$ and $M = 400$.

If one wants to achieve a better approximation for the centre crack, the value of m must be increased. A more elongated ellipse is obtained with $m = 0.99$. However, a satisfactory convergence is achieved with $M > 700$ as shown in Figure 2.2.

Equations (1.59)–(1.62) can be simplified by taking (2.13) into account. It follows from (2.13) that

$$\text{Im}c_k \equiv 0 \quad (2.15)$$

and

$$c_{l+k-1} = 0 \quad \forall l + k - 1 < -1 \quad (2.16)$$

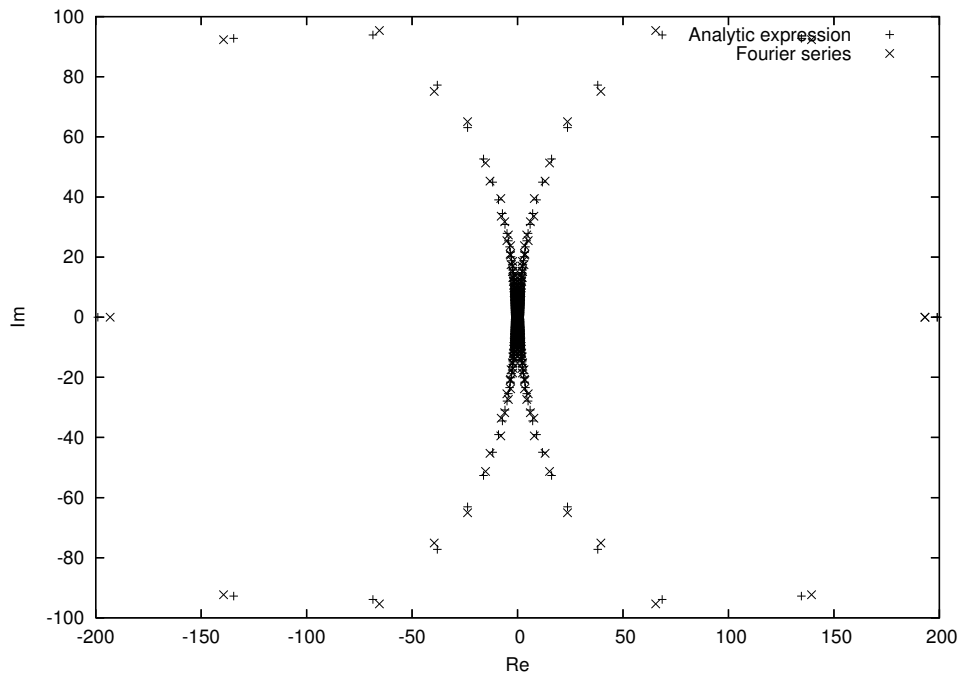


Figure 2.2: Comparison of (2.7) and (2.14) for $m = 0.99$ and $M = 700$.

or

$$c_{l+k-1} = 0 \quad \forall l < -k. \quad (2.17)$$

Therefore the summation in (1.59)–(1.62) should start at $l = -k$:

$$\begin{aligned}
A_k &= \varkappa \operatorname{Re} \zeta^k - k \operatorname{Re} \Omega \operatorname{Re} \zeta^{k-1} - k \operatorname{Im} \Omega \operatorname{Im} \zeta^{k-1} + \operatorname{Re} \zeta^{-k} + \\
&\quad + k \sum_{l=-k}^M \operatorname{Re} c_{l+k-1} \operatorname{Re} \zeta^{-l}
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
B_k &= -\varkappa \operatorname{Im} \zeta^k + k \operatorname{Re} \Omega \operatorname{Im} \zeta^{k-1} - k \operatorname{Im} \Omega \operatorname{Re} \zeta^{k-1} + \operatorname{Im} \zeta^{-k} - \\
&\quad - k \sum_{l=-k}^M \operatorname{Re} c_{l+k-1} \operatorname{Im} \zeta^{-l}
\end{aligned} \tag{2.19}$$

$$\begin{aligned}
C_k &= \varkappa \operatorname{Im} \zeta^k + k \operatorname{Re} \Omega \operatorname{Im} \zeta^{k-1} - k \operatorname{Im} \Omega \operatorname{Re} \zeta^{k-1} - \operatorname{Im} \zeta^{-k} - \\
&\quad - k \sum_{l=-k}^M \operatorname{Re} c_{l+k-1} \operatorname{Im} \zeta^{-l}
\end{aligned} \tag{2.20}$$

$$\begin{aligned}
D_k &= \varkappa \operatorname{Re} \zeta^k + k \operatorname{Re} \Omega \operatorname{Re} \zeta^{k-1} + k \operatorname{Im} \Omega \operatorname{Im} \zeta^{k-1} + \operatorname{Re} \zeta^{-k} - \\
&\quad - k \sum_{l=-k}^M \operatorname{Re} c_{l+k-1} \operatorname{Re} \zeta^{-l}
\end{aligned} \tag{2.21}$$

Chapter 3

Stress Intensity Factors

It was shown by Sih et al. (1962) that the complex stress intensity factor

$$K = K_I - iK_{II} \quad (3.1)$$

can be found as

$$K = 2\sqrt{2\pi} \lim_{\zeta \rightarrow 1} \phi'(\zeta) \frac{\sqrt{\omega(\zeta) - \omega(1)}}{\omega'(\zeta)}. \quad (3.2)$$

Note $\sqrt{\pi}$ factor which does not appear in Sih et al. (1962).

Using (2.1) with $m = 1$ and (1.55) one obtains:

$$K = 2\sqrt{\frac{\pi}{a}} \sum_{k=-N}^N ka_k, \quad (3.3)$$

where $a = 2R$ is a half length of a centre crack in an infinite plane. This value is obtained from (2.1) by noting that $z = a$ corresponds to $\zeta = 1$.

Chapter 4

Westergaard crack tip fields

The displacement and stress crack tip fields under mixed mode (I + II) loading can be simulated using the following equations:

$$\sigma_{11} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) - \frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \left(2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right) \quad (4.1)$$

$$\sigma_{22} = \frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) + \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \quad (4.2)$$

$$\sigma_{12} = \frac{K_I}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} + \frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) \quad (4.3)$$

$$u_{11} = \frac{K_I}{\mu} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left(1 - 2\delta + \sin^2 \frac{\theta}{2} \right) + \frac{K_{II}}{\mu} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left(2 - 2\delta + \cos^2 \frac{\theta}{2} \right) \quad (4.4)$$

$$u_{22} = \frac{K_I}{\mu} \sqrt{\frac{r}{2\pi}} \sin \frac{\theta}{2} \left(1 - 2\delta + \sin^2 \frac{\theta}{2} \right) + \frac{K_{II}}{\mu} \sqrt{\frac{r}{2\pi}} \cos \frac{\theta}{2} \left(2\delta - 1 + \sin^2 \frac{\theta}{2} \right) \quad (4.5)$$

where

$$\delta = \begin{cases} \nu; & \text{plane strain} \\ \frac{\nu}{1+\nu}; & \text{plane stress} \end{cases} \quad (4.6)$$

Chapter 5

Verification

5.1 Linear system with unique solution

To verify the expressions of Chapter 1 a FORTRAN 90 program, `dread.f90`, was written for calculating the SIFs from simulated displacement fields, obtained as described in Chapter 4.

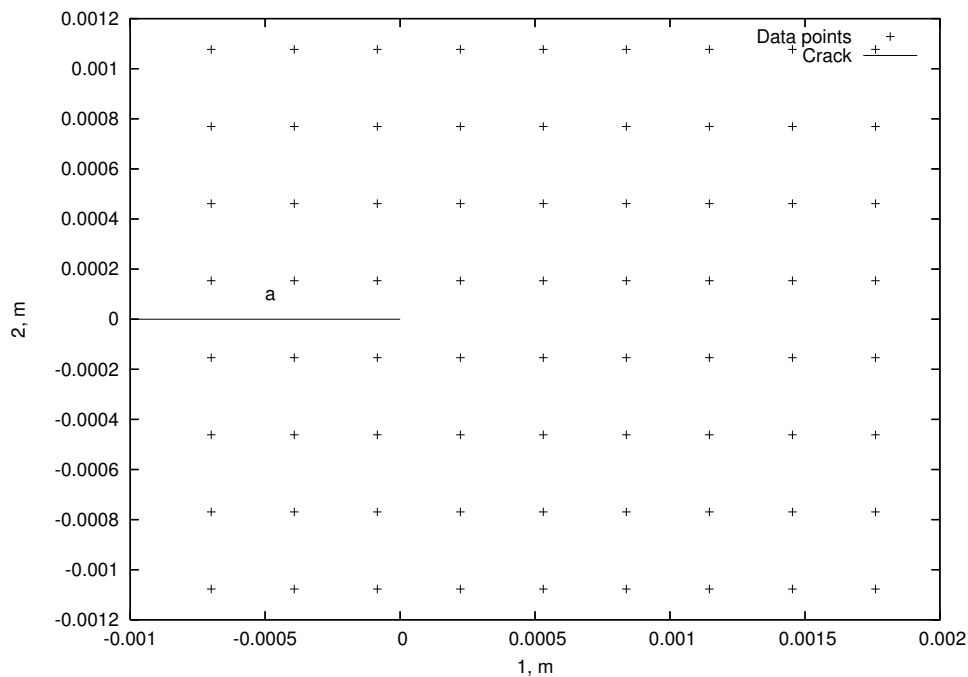


Figure 5.1: Data points for which displacement data was generated.

A crack with $a = 1$ mm was assumed. It was loaded with $K_I = K_{II} = 50$ $\text{MPa}\sqrt{\text{m}}$. The material properties used were: the Young's modulus, $E =$

2×10^5 MPa, and the Poisson's ratio, $\nu = 0.3$.

Displacement fields, u_{11} and u_{22} , were obtained using (4.4) and (4.5) at 105 points located as shown in Figure 5.1.

The mapping plane coordinates, ζ , of the data points were calculated using (2.2). These points are shown in Figure 5.2.

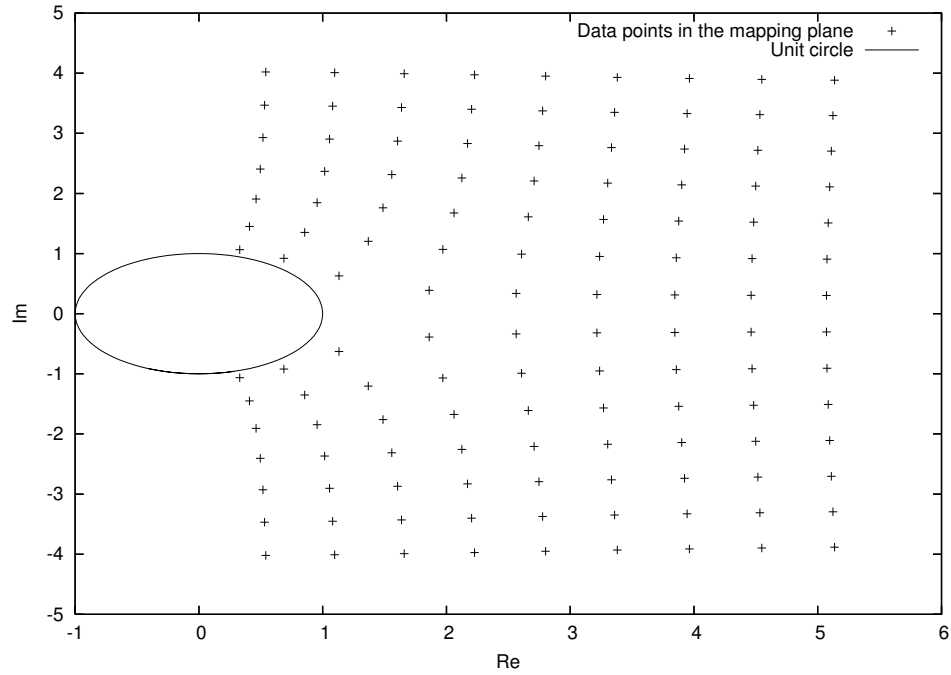


Figure 5.2: Data points in the mapping plane.

N was chosen so that $p = 2N + 1$. Accordingly the number of unknowns (1.67) is equal to number of equations (1.68). Matrix \mathbf{E} was created as in (1.66) where A_k^j , B_k^j , C_k^j and D_k^j are given by (2.18)–(2.21). The system (1.65) is solved by LU decomposition and the SIFs are calculated as in (3.3).

The results are shown in Table 5.1, where m is the shape parameter of the mapping function (2.1) and M is the series limit in (2.18)–(2.21). The algorithm gives a good solution even for m as small as 0.9. As $m \rightarrow 1$, $m = 0.99$ and $m = 0.992$, the accuracy of solution is improving. However, the number of terms in the series (2.18)–(2.21) is increasing rapidly and so does the computation time. Further increase in m , $m = 0.994$, gives little improvement in accuracy, and at $m = 0.996$ the accuracy actually is worse. This is probably caused by the accumulation of the round-off errors for the large values of M .

m	M	Calculated K_I and K_{II} , MPa \sqrt{m}	Error, %
0.9	100	50.96 -51.05	-1.9 -2.1
0.99	800	49.97 50.12	0.05 -0.25
0.992	1000	50.005 50.016	-0.01 -0.03
0.994	1600	50.042 50.004	-0.084 -0.009
0.995	2000	49.930 49.886	0.14 0.23

Table 5.1: The calculated SIFs for applied $K_I = K_{II} = 50 \text{ MPa}\sqrt{m}$.

5.2 Overdetermined system

5.2.1 Crack length

In principle the SIF solution is independent of the crack length, a .

Indeed, as follows from (4.4) and (4.5), the displacement fields are only dependent on the SIFs. Infinite number of combinations of loading and crack length produce identical SIF values. Thus given displacement fields nothing can be said about the crack length.

Mathematically the crack length affects the scaling factor of the mapping function, R , eqn (2.1), i.e. where on the mapping plane the displacement points are located. This scaling is compensated by the presence of the crack length in (3.3).

The obvious conclusion is that in this case it should be possible to completely exclude the crack length from the analysis. I have not been able to do so.

However, in practice, the crack length has an effect on the resulting SIFs due to limited machine precision. The following is an example of such influence.

A crack tip loaded at $K_I = 10 \text{ MPa}\sqrt{m}$ and $K_{II} = 35 \text{ MPa}\sqrt{m}$ is analysed. The displacements have been generated at the 126 points shown in Figure 5.3.

The Fourier series limit for the analytical functions is taken $N = 50$. Accordingly there are $2N + 1 = 101$ unknowns and the system is overdetermined. The solution is searched in the linear least squares sense using the QR decomposition. A FORTRAN 90 program `dread1.1.f90` was written that

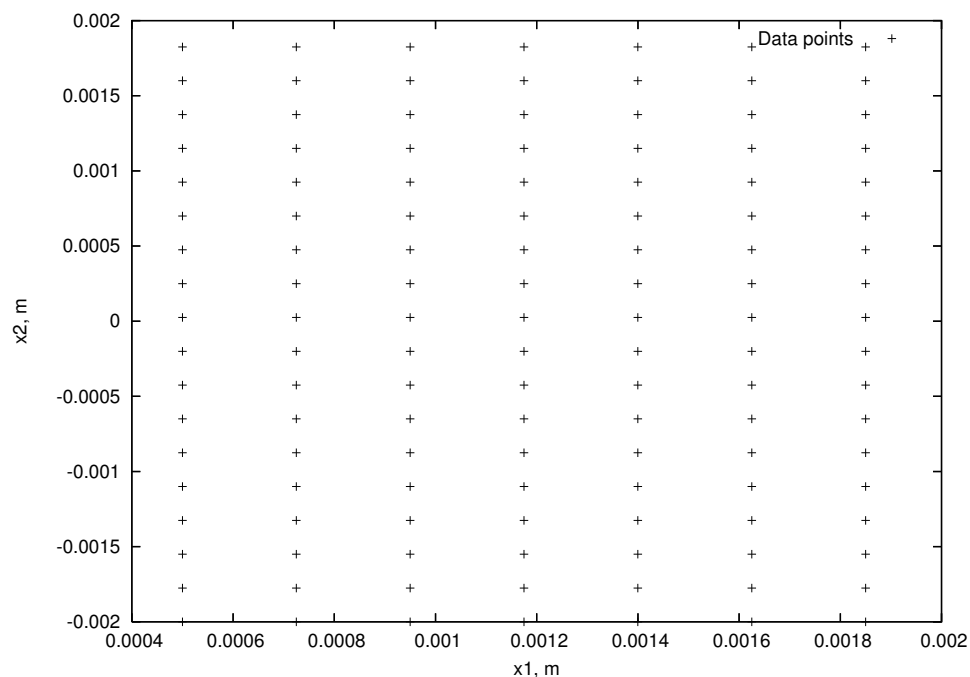


Figure 5.3: Data points in the physical plane.

implemented this algorithm.

The shape of the mapping function, m , is taken as $m = 0.992$. The Fourier series limit in (2.18)–(2.21), M , is then chosen according to the results given in Table 5.1, $M = 1000$.

The SIF solution was obtained with several values of a . The results are shown in Table 5.2 and in Figure 5.4.

These results show that the most accurate SIF solution is obtained when a is of the order of the longest distance from the crack tip to a data point. In this example this distance is approximately 2 mm, see Figure 5.3. The smallest error was obtained with $a = 1$ mm. When a is moving away from this value the error increases, see Figure 5.4 or Table 5.2. The error grows much faster when a is decreasing than when a is increasing.

To understand such error behaviour one has to look at the data point locations in the mapping plane. As was said earlier, these depend on a , according to (2.2), where $R = a/(1 + m)$.

If a is very small, then the data points in the mapping plane are so far from the unit circle, that the displacements there are very slightly affected by the singular field. Because the machine precision is limited, the precise SIF solution is hard to obtain, or not possible at all. For $a = 1 \times 10^{-5}$ m the data points are so far from the unit circle, see Figure 5.5, that the SIF error

a , m	Calculated K_I and K_{II} , MPa \sqrt{m}	Error, %
1×10^{-3}	10.00	-0.004
	35.02	-0.053
1×10^{-2}	10.00	-0.008
	35.03	-0.098
1×10^{-4}	10.99	-9.940
	38.83	-10.93
1×10^{-1}	10.00	-0.037
	35.17	-0.489
1×10^{-5}	7.626	31.13
	8.705	302.1
1	10.08	-0.784
	36.00	-2.857
1×10^1	10.57	-5.710
	40.51	-15.75
1×10^2	12.91	-29.13
	52.50	-49.99
1×10^3	30.99	-209.9
	129.1	-268.8

Table 5.2: The calculated SIFs for applied $K_I = 10 \text{ MPa}\sqrt{m}$ and $K_{II} = 35 \text{ MPa}\sqrt{m}$.

is higher than 300 %. With a increasing to $a = 1 \times 10^{-4}$ m the data points move closer to the unit circle, see Figure 5.6, and the error is about 11 %.

When a is comparable with the maximum distance from a data point to the crack tip, the data points in the mapping plane are located close to the unit circle, see Figure 5.7. At the same time the points are well defined, that is the distance between the points is long enough so that the mapping plane coordinates of each points are unique to many significant digits. Hence a very small round-off error, approximately 0.05 %

With increasing a the mapping plane coordinates of the data points move closer to each other, see Figures 5.8 – 5.12. As a consequence, the number of unique significant digits used to store the coordinates is decreasing and so is the SIF precision.

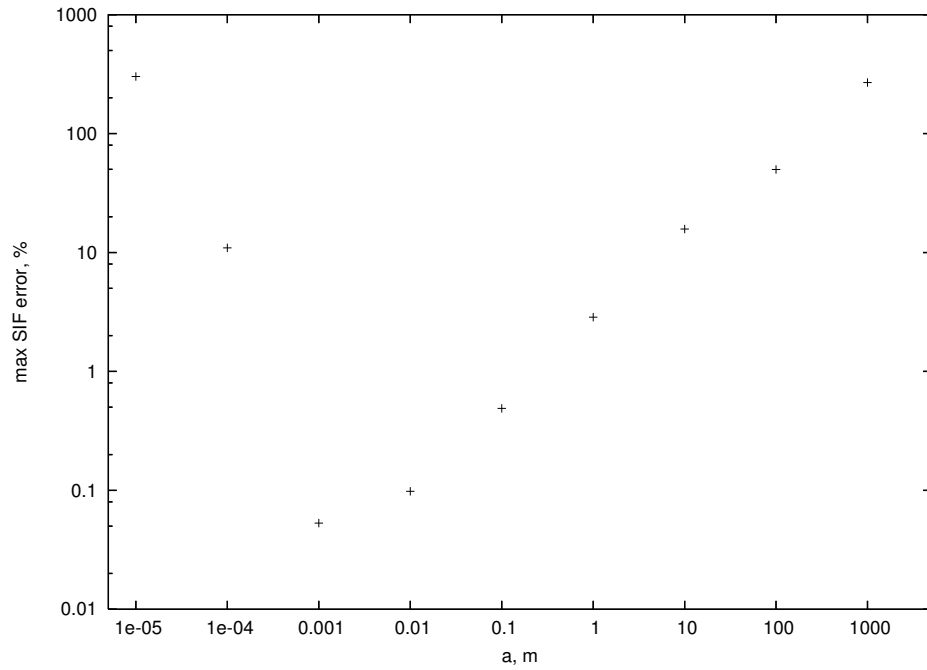


Figure 5.4: Maximum absolute value of the calculated SIF error is plotted against the crack length, a . The data is taken from Table 5.2. Note the logarithmic scale for both a and SIF error.

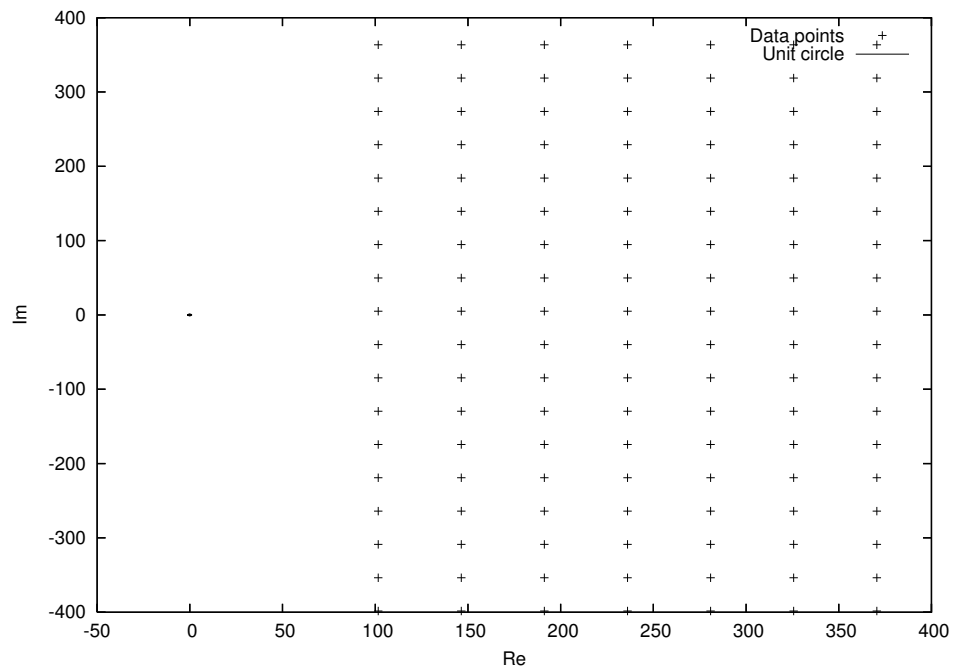


Figure 5.5: Data point locations in the mapping plane for $a = 1 \times 10^{-5}$ m.

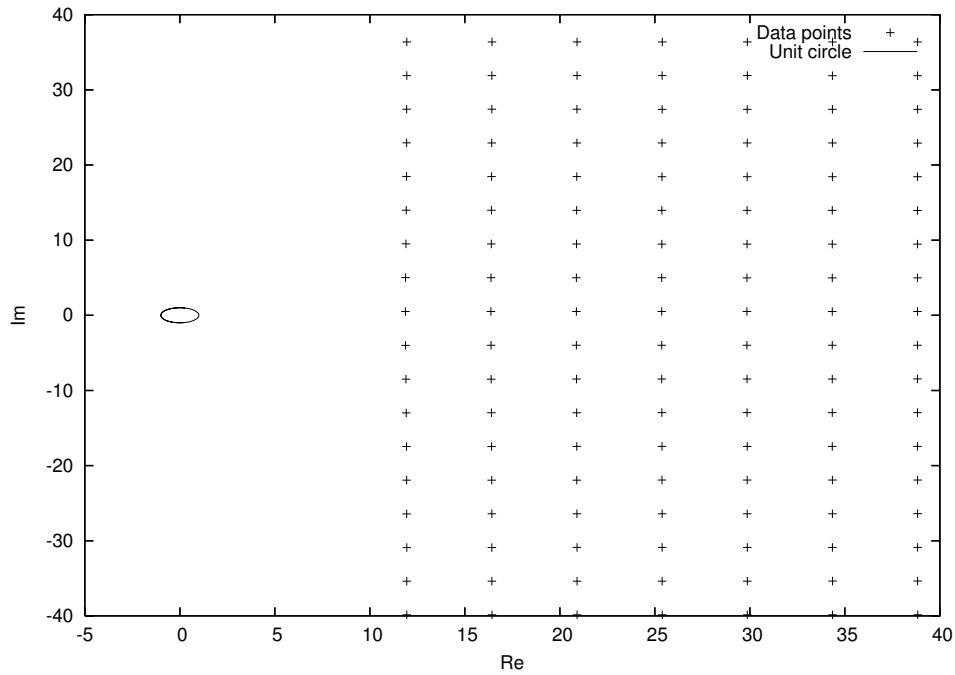


Figure 5.6: Data point locations in the mapping plane for $a = 1 \times 10^{-4}$ m.

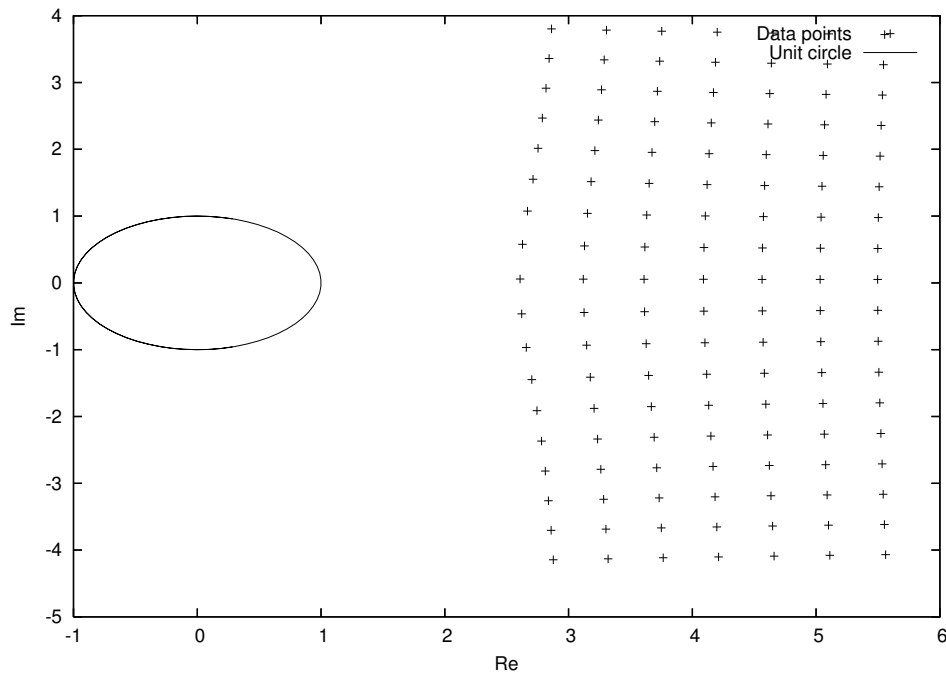


Figure 5.7: Data point locations in the mapping plane for $a = 1 \times 10^{-3}$ m.

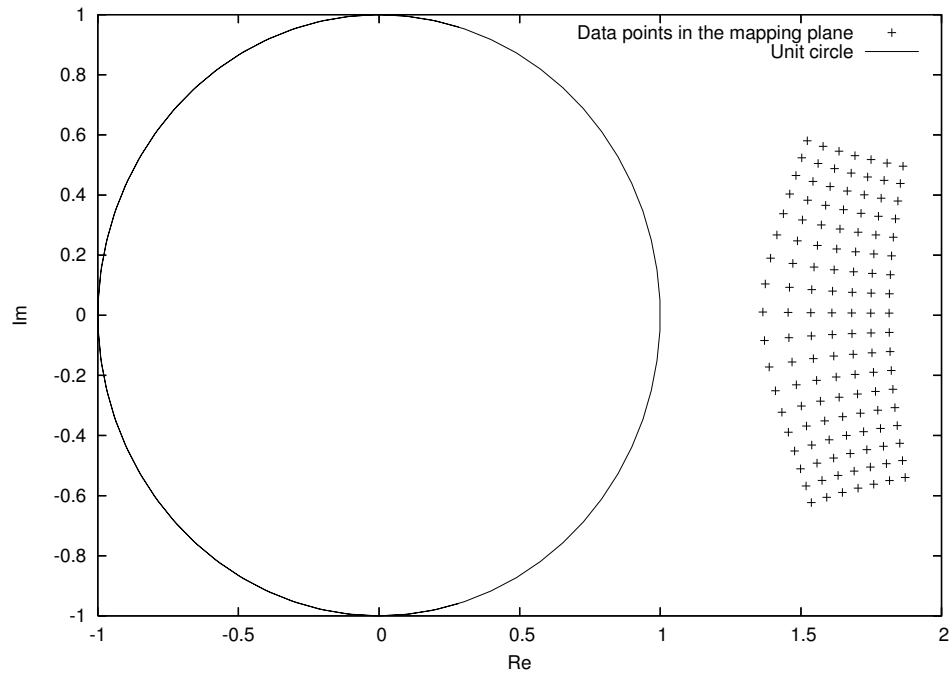


Figure 5.8: Data point locations in the mapping plane for $a = 1 \times 10^{-2}$ m.

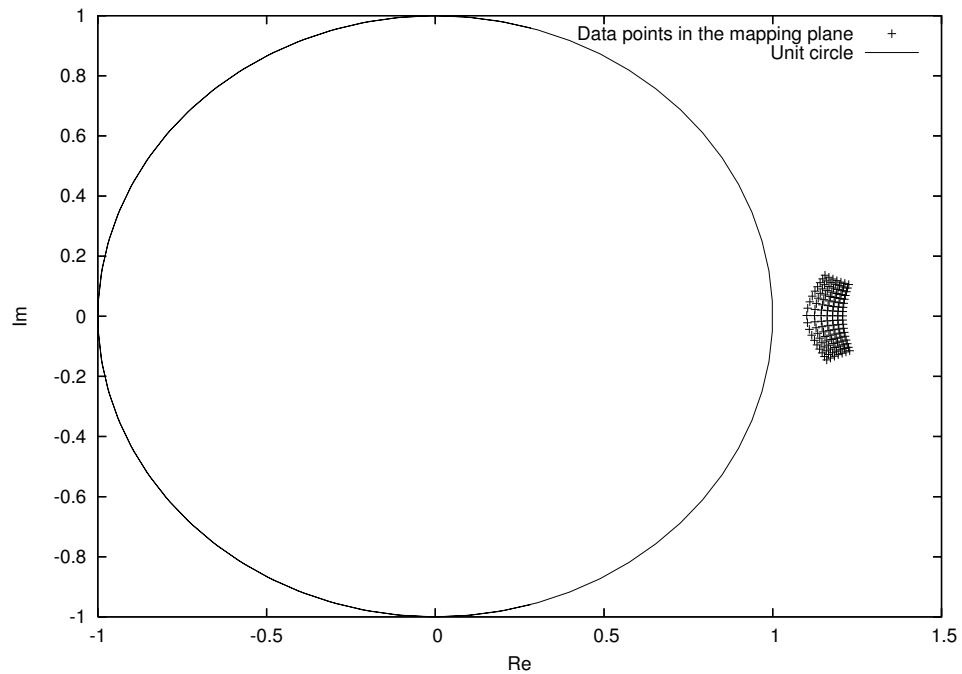
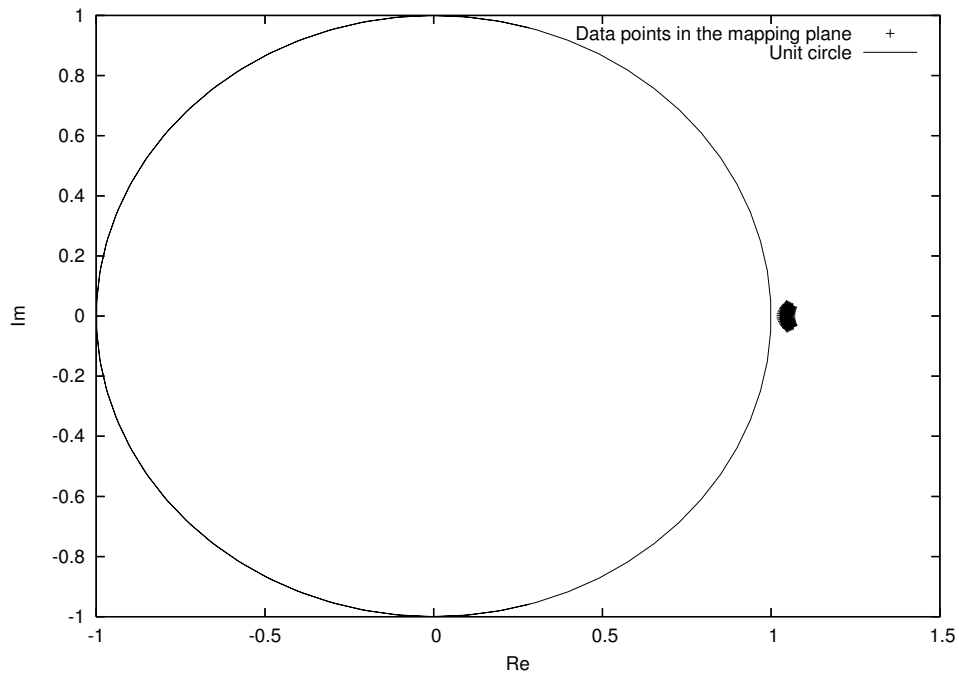
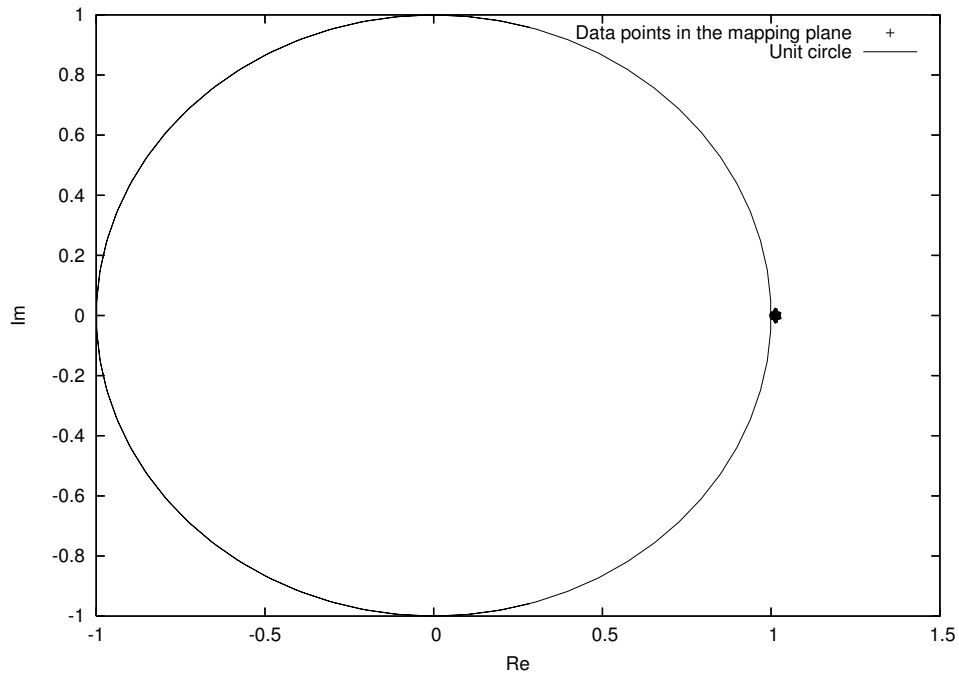


Figure 5.9: Data point locations in the mapping plane for $a = 1 \times 10^{-1}$ m.

Figure 5.10: Data point locations in the mapping plane for $a = 1$ m.Figure 5.11: Data point locations in the mapping plane for $a = 1 \times 10^1$ m.

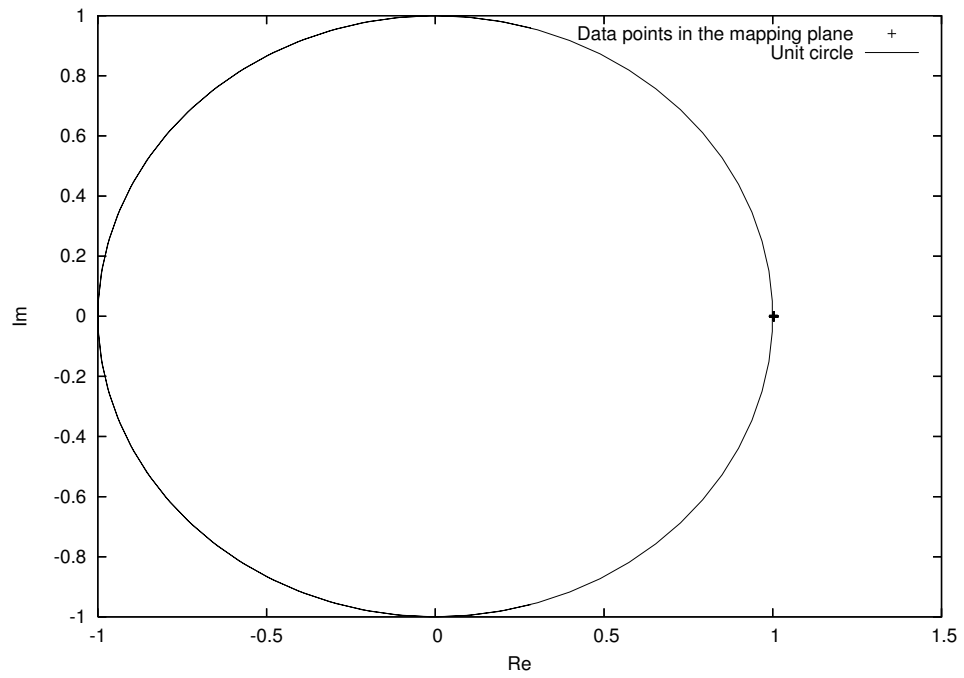


Figure 5.12: Data point locations in the mapping plane for $a = 1 \times 10^2$ m.

Chapter 6

Genetic Algorithm

If the crack tip location is not known then system (1.63)-(1.64) becomes non-linear and has to be solved by some minimization method. It is proposed to use Genetic Algorithm. The method is based on minimization of the objective function. If we write the system of equations in indicial notation as:

$$f_i(x_j) = 0 \quad (6.1)$$

where $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$, n is the number of equations and m is the number of variables, then the objective function can be constructed as follows:

$$F = \sqrt{\sum_1^n f_i(x_j^*) f_i(x_j^*)} \quad (6.2)$$

where x_i^* is an approximate solution to (6.1). In other words the objective function is a square root of the sum of squares of residuals of all functions f_i .

Let's designate the unknown 'true' crack tip location as $z_0 = x_1^0 + ix_2^0$. Then the data points will have the following coordinates relative to the true crack tip:

$$z_j = z_j^{\text{measured}} - z_0 \quad (6.3)$$

Using (1.63)-(1.64) the objective function can be written as follows:

$$F = \left\{ \sum_{j=1}^p \left(\left[\sum_{k=-N}^N (A_k^j \alpha_k + B_k^j \beta_k) - 2\mu w_1^j \right]^2 + \left[\sum_{k=-N}^N (C_k^j \alpha_k + D_k^j \beta_k) - 2\mu w_2^j \right]^2 \right) \right\}^{1/2} \quad (6.4)$$

where A_k^j, B_k^j, C_k^j and D_k^j are calculated from (1.59)-(1.62) for ζ_j which in turn is calculated using (2.2) for z_j calculated as in (6.3).

Bibliography

Golub, G. H. and C. F. Van Loan (1996), *Matrix Computations*, 3 edn, The John Hopkins University Press.

Muskhelishvili, N. I. (1977), *Some Basic Problems of the Mathematical Theory of Elasticity*, 4th edn, Noordhoff International.

Press, W. H., S. A. Teukolsky, W. T. Vetterling and B. P. Flannery (1996), *Numerical Recipes in Fortran 90: the Art of Parallel Scientific Computing*, Vol. 2 of *Fortran Numerical Recipes*, 2 edn, Cambridge University Press.

Sih, G. C., P. C. Paris and F. Erdogan (1962), ‘Crack-tip, stress-intensity factors for plane extension and plate bending problems’, *Journal of Applied Mechanics* **29**, 306–312.

Sokolnikoff, I. S. and E. S. Sokolnikoff (1941), *Higher Mathematics for Engineers and Physicists*, 2 edn, McGraw-Hill.